MUMBAI UNIVERSITY

SEMESTER – II

APPLIED MATHEMATICS - II

QUESTION PAPER – MAY 2019

Q.1

a) Evaluate $\int_0^{\infty} y^4 e^{-y^6} dy$

Solution :

Let I = $\int_0^{\infty} y^4 e^{-y^6} dy$ and $y^6 = t$

$$y = t^{\frac{1}{6}}$$

$$dy = \frac{dt}{6t^{\frac{5}{6}}}$$

When y=0, t=0 and when y=∞, t=∞

Now,

I = $\int_0^{\infty} y^4 e^{-y^6} dy$

= $\int_0^{\infty} \left(t^{\frac{1}{6}}\right)^4 e^{-t} \frac{dt}{6y^{\frac{5}{6}}}$

= $\int_0^{\infty} t^{\frac{-1}{6}} e^{-t} dt$

= $\Gamma\left(\frac{5}{6}\right)$

$$\boxed{\int_0^{\infty} y^4 e^{-y^6} dy = \Gamma\left(\frac{5}{6}\right)}$$

b) Find the circumference of a circle of radius r by using parametric equations of the circle x=rcosɵ, y= rsinɵ.

Solution :

For a circle with radius r and parametric equations x=rcosɵ and y= rsinɵ,

Circumference, c = $\int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

= $\int_0^{2\pi} \sqrt{(-rsin\theta)^2 + (rcos\theta)^2} d\theta$

= $\int_0^{2\pi} r\sqrt{sin^2\theta + cos^2\theta} d\theta$

= r$\int_0^{2\pi} d\theta$

= r$[\theta]_0^{2\pi}$

$\boxed{\text{c} = 2\pi r}$

c) Solve $(D^2 + D - 6)y = e^{4x}$

Solution :

The auxiliary equation is $D^2 + D - 6 = 0$.

$(D-2)(D+3) = 0$

$D = 2, -3$

Complementary Function, $\boxed{C.F. = c_1e^{2x} + c_2e^{-3x}}$

Particular Integral, P.I. $= \frac{1}{(D-2)(D+3)} e^{4x}$

$= \frac{1}{(4-2)(4+3)} e^{4x}$

$= \frac{1}{2 \times 7} e^{4x}$

$$\boxed{P.I. = \frac{e^{4x}}{14}}$$

The complete solution is y = C.F. + P.I.

$$\boxed{y = c_1e^{2x} + c_2e^{-3x} + \frac{e^{4x}}{14}}$$

d) Evaluate $\int_0^1 \int_{x^2}^{x} xy(x^2 + y^2)dydx$

Solution :

Let I = $\int_0^1 \int_{x^2}^{x} xy(x^2 + y^2)dydx$

I = $\int_0^1 \int_{x^2}^{x} x^3y + y^3xdydx$

Integrating w.r.t y,

I = $\int_0^1 \left[x^3\frac{y^2}{2} + \frac{y^4}{4}x\right]_{x^2}^{x} dx$

I = $\int_0^1 x^3\frac{x^2}{2} + \frac{x^4}{4}x - x^3\frac{(x^2)^2}{2} - \frac{(x^2)^4}{4}x\ dx$

I = $\int_0^1 \frac{x^5}{2} + \frac{x^5}{4} - \frac{x^7}{2} - \frac{x^9}{4}\ dx$

I = $\int_0^1 \frac{3x^5}{4} - \frac{x^7}{2} - \frac{x^9}{4}\ dx$

Integrating w.r.t x,

I = $\left[\frac{3x^6}{4\ X\ 6} - \frac{x^8}{2\ X\ 8} - \frac{x^{10}}{4\ X\ 10}\right]_0^1$

I = $\frac{3}{24} - \frac{1}{16} - \frac{1}{40}$

I = $\frac{3}{80}$

$$\boxed{\int_0^1 \int_{x^2}^{x} xy(x^2 + y^2)dydx = \frac{3}{80}}$$

e) Solve (tany + x)dx + (xsec²y – 3y)dy =0

Solution :

Comparing the equation (tany + x)dx + (xsec^{2}y – 3y)dy =0 with Mdx + Ndy = 0,

M = tany + x
N = xsec^{2}y – 3y

$\frac{\partial M}{\partial y} = sec^2 y \quad \frac{\partial N}{\partial x} = sec^2 y$

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given D.E. is exact

$\int Mdx = \int (tany + x)\,dx = xtany + \frac{x^2}{2}$

$\int (Terms\ in\ N\ free\ from\ x)dy = \int -3ydy = \frac{-3y^2}{2}$

Solution,
$\int Mdx + \int (Terms\ in\ N\ free\ from\ x)dy = c$

$$\boxed{xtany + \frac{x^2}{2} - \frac{3y^2}{2} = c}$$

f) Solve $\frac{dy}{dx} = 1 + xy$ with initial condition $x_0 = 0$, $y_0 = 0.2$ by Euler's method. Find the approximate value of y at x = 0.4 with h = 0.1

Solution :

Since $f(x,y) = 1 + xy$, $f(x_0,y_0) = 1 + (0 \times 0.2) = 1$

At $x_1 = 0.1$, $y_1 = y_0 + h\, f(x_0,y_0) = 0.2 + \{0.1 \times [1 + (0 \times 0.2)]\} = 0.2 + 0.1 = 0.3$

At $x_2 = 0.2$, $y_2 = y_1 + h\, f(x_1,y_1) = 0.3 + \{0.1 \times [1 + (0.1 \times 0.3)]\} = 0.3 + 0.103 = 0.403$

At $x_3 = 0.3$, $y_3 = y_2 + h\, f(x_2,y_2) = 0.403 + \{0.1 \times [1 + (0 \times 0.2)]\} = 0.2 + 0.1 = 0.511$

At $x_4 = 0.4$, $y_4 = y_3 + h\, f(x_3,y_3) = 0.2 + \{0.1 \times [1 + (0 \times 0.2)]\} = 0.2 + 0.1 = 0.6263$

$\boxed{\text{At } x = 0.4,\ y = 0.6263}$

Q.2
a) Solve $(D^2 - 4D + 3)y = e^x\cos 2x + x^2$

<u>Solution</u> :

The auxiliary equation is $D^2 - 4D + 3$

$(D-3)(D-1) = 0$

$D = 3, 1$

Complementary Function, $\boxed{C.F. = c_1e^{3x} + c_2e^x}$

Particular Integral , P.I. $= \frac{1}{(D-3)(D-1)}(e^x\cos 2x + x^2)$

$= \frac{1}{(D-3)(D-1)} e^x\cos 2x + \frac{1}{(D-3)(D-1)} x^2$ Type equation here.

$= e^x\frac{1}{(D+1-3)(D+1-1)}\cos 2x + 3\left(1 - \frac{D}{3}\right)^{-1}(1 - D)^{-1} x^2$

$= e^x\frac{1}{(D-2)(D)}\cos 2x + 3\left(1 + \frac{D}{3} + \frac{D^2}{9}\right)(1 + D + D^2)\, x^2$

$= e^x\frac{1}{D^2-2D}\cos 2x + 3\left(1 + \frac{D}{3} + \frac{D^2}{9}\right)(x^2 + 2x + 2)$

$= e^x\frac{1}{-4-2D}\cos 2x + 3\left(x^2 + 2x + 2 + \frac{2x}{3} + \frac{2}{3} + \frac{2}{9}\right)$

$= -\frac{e^x}{2}\frac{1}{D+2}\cos 2x + 3x^2 + 8x + \frac{26}{3}$

$= -\frac{e^x}{2}\frac{D-2}{D^2-4}\cos 2x + 3x^2 + 8x + \frac{26}{3}$

$= -\frac{e^x}{2}\frac{D-2}{-4-4}\cos 2x + 3x^2 + 8x + \frac{26}{3}$

$= \frac{e^x}{16}(-2\sin 2x - 2\cos 2x) + 3x^2 + 8x + \frac{26}{3}$

$= \frac{-e^x}{8}(\sin 2x + \cos 2x) + 3x^2 + 8x + \frac{26}{3}$

$= \frac{-e^x}{8}\sqrt{2}\cos\left(2x - \frac{\pi}{4}\right) + 3x^2 + 8x + \frac{26}{3}$

The complete solution is y = C.F. + P.I.

$$\boxed{y = c_1e^{3x} + c_2e^x - \frac{e^x}{8}\sqrt{2}\cos\left(2x - \frac{\pi}{4}\right) + 3x^2 + 8x + \frac{26}{3}}$$

b) Show that $\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)} dx = \frac{\pi}{2} log(1+a)$

Solution :

$$I(a) = \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)} dx$$

By the rule of differentiation under integral sign we have, differentiating w.r.t a,

$$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}\left(\frac{\tan^{-1} ax}{x(1+x^2)}\right) dx$$

$$= \int_0^\infty \left(\frac{x}{1+a^2x^2} \cdot \frac{1}{x(1+x^2)}\right) dx$$

$$= \int_0^\infty \left(\frac{1}{(1+a^2x^2)} \cdot \frac{1}{(1+x^2)}\right) dx$$

$$= \frac{1}{1-a^2}\int_0^\infty \left(\frac{1}{(1+x^2)} - \frac{a^2}{(1+a^2x^2)}\right) dx$$

$$= \frac{1}{1-a^2}\left[\tan^{-1} x - a\tan^{-1} ax\right]_0^\infty$$

$$= \frac{1}{1-a^2}\left(\frac{\pi}{2} - a\frac{\pi}{2}\right)$$

$$\frac{dI}{da} = \frac{\pi}{2} \cdot \frac{1}{1+a}$$

Integrating both sides w.r.t a,

$$I = \int \frac{\pi}{2} \cdot \frac{1}{1+a}\, da$$

$$I = \frac{\pi}{2}\log(1+a)$$

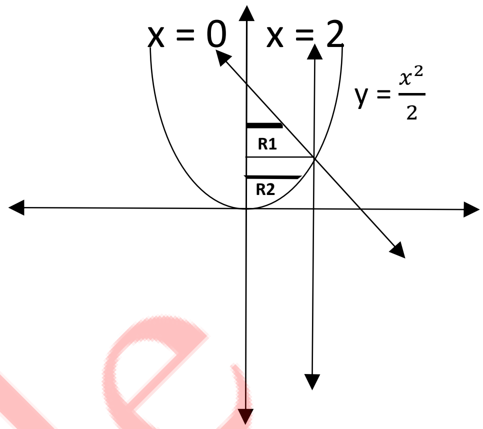
$$\boxed{\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)} dx = \frac{\pi}{2} log(1+a)}$$

c) Change the order of integration and evaluate $\int_0^2 \int_{\frac{x^2}{2}}^{4-x} xy\,dy\,dx$

Solution : Let I = $\int_0^2 \int_{\frac{x^2}{2}}^{4-x} xy\,dy\,dx$

x = 2, x = 0, y = 4-x, y = $\frac{x^2}{2}$

After changing the order of integration, we get two parts, R1 and R2 of the common region where the limits of the variables do not change.



In R1, x varies from 0 to 4-y and varies from 2 to 4

In R2, x varies from 0 to $\sqrt{2y}$ and varies from 0 to 2

$$I = \int_2^4 \int_0^{4-y} xy\,dx\,dy + \int_0^2 \int_0^{\sqrt{2y}} xy\,dx\,dy$$

$$I = \int_2^4 y\left[\frac{x^2}{2}\right]_0^{4-y} dy + \int_0^2 y\left[\frac{x^2}{2}\right]_0^{\sqrt{2y}} dy$$

$$I = \int_2^4 y\,\frac{(4-y)^2}{2}\,dy + \int_0^2 y\,\frac{(\sqrt{2y})^2}{2}\,dy$$

$$I = \int_2^4 y\,\frac{(16-8y+y^2)}{2}\,dy + \int_0^2 y\,\frac{2y}{2}\,dy$$

$$I = \frac{1}{2}\int_2^4 (16y - 8y^2 + y^3)\,dy + \int_0^2 y^2\,dy$$

$$I = \frac{1}{2}\left[8y^2 - \frac{8y^3}{3} + \frac{y^4}{4}\right]_2^4 + \left[\frac{y^3}{3}\right]_0^2$$

$$I = \frac{1}{2}\left[8.4^2 - \frac{8.4^3}{3} + \frac{4^4}{4} - 8.2^2 + \frac{8.2^3}{3} - \frac{2^4}{4}\right] + \left[\frac{2^3}{3}\right]$$

$$I = \frac{10}{3} + \frac{8}{3}$$

I = 6

$$\boxed{\int_0^2 \int_{\frac{x^2}{2}}^{4-x} xy\,dy\,dx = 6}$$

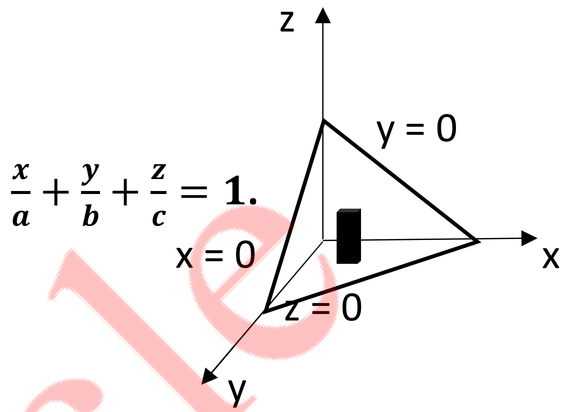
Q.3

a) Evaluate $\iiint x^2yzdxdydz$ throughout the volume bounded by the planes x = 0, y = 0, z = 0 and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

Solution :

Let x = au, y = bv, z = cw

dx = a.du, dy = b.dv, dz = c.dw



I = $\iiint x^2yzdxdydz$

I = $\iiint (au)^2.bv.cw.a.du.b.dv.c.dw$

I = $a^3b^2c^2\iiint u^2vwdudv.dw$

The planes will become, u = 0, v = 0, w = 0 and u + v + w = 0.

If we consider an elementary cuboid, on this cuboid,

w varies from 0 to 1 – u – v

v varies from 0 to 1 – u

u varies from 0 to 1

I = $a^3b^2c^2\int_0^1\int_0^{1-u}\int_0^{1-u-v}u^2vwdwdvdu$

I = $a^3b^2c^2\int_0^1\int_0^{1-u}u^2v\left[\frac{w^2}{2}\right]_0^{1-u-v}dvdu$

I = $a^3b^2c^2\int_0^1\int_0^{1-u}u^2v\frac{(1-u-v)^2}{2}dvdu$

I = $a^3b^2c^2\int_0^1\int_0^{1-u}u^2v\frac{[(1-u)^2-2(1-u)v+v^2]}{2}dvdu$

I = $a^3b^2c^2\int_0^1\int_0^{1-u}u^2\frac{[(1-u)^2v-2(1-u)v^2+v^3]}{2}dvdu$

I = $\frac{a^3b^2c^2}{2}\int_0^1u^2[(1-u)^2\frac{v^2}{2}-2(1-u)\frac{v^3}{3}+\frac{v^4}{4}]_0^{1-u}du$

I = $\frac{a^3b^2c^2}{2}\int_0^1u^2[\frac{(1-u)^4}{2}-\frac{2(1-u)^4}{3}+\frac{(1-u)^4}{4}]du$

I = $\frac{a^3b^2c^2}{2}\int_0^1\frac{u^2(1-u)^4}{12}du$

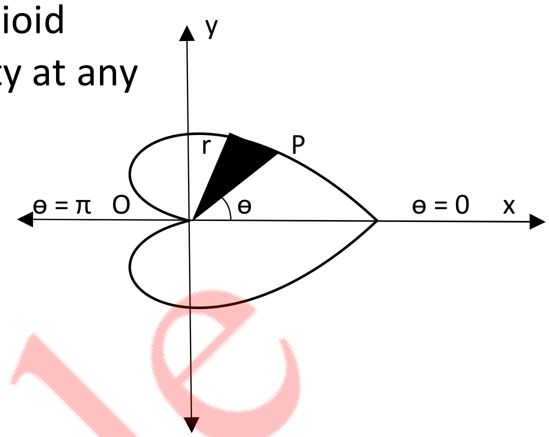
I = $\frac{a^3b^2c^2}{24}$ β(3,5)

I = $\frac{a^3b^2c^2}{24}$ x $\frac{2!4!}{7!}$

I = $\frac{a^3b^2c^2}{2520}$

$\iiint x^2yz\,dxdydz = \frac{a^3b^2c^2}{2520}$ throughout the volume bounded by the planes x = 0, y = 0, z = 0 and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

b) Find the mass of the lamina of a cardioid r = a(1 + cosɵ). If the density at any point varies as the square of its distance from its axis of symmetry.



Solution : Let P(r,ɵ) be any point on the given cardioid
The distance of P from the axis is r sinɵ. The density at any point P(r,ɵ) is ρ = k $r^2 sin^2\theta$.
Consider a radial strip in the first quadrant.
On this strip, r varies from 0 to a(1 + cosɵ) and
ɵ varies from 0 to π.

Mass of the lamina,

$$= 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} (k\, r^2 \sin^2\theta)\, r\,dr\,d\theta$$

$$= 2k\int_0^{\pi} \sin^2\theta \left[\frac{r^4}{4}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{ka^4}{2}\int_0^{\pi} \sin^2\theta\,(1+\cos\theta)^4 d\theta$$

$$= \frac{ka^4}{2}\int_0^{\pi} \left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2 \left(2\cos^2\frac{\theta}{2}\right)^4 d\theta$$

$$= 32\,ka^4 \int_0^{\pi} \sin^2\frac{\theta}{2}\cos^{10}\frac{\theta}{2}\,d\theta$$

$$= 64\,ka^4 \int_0^{\pi} \sin^2 t\,\cos^{10} t\,dt \qquad \left[\frac{\theta}{2} = t\right]$$

$$= 64\,ka^4\,\frac{1.9.7.5.3.1}{12.10.8.6.4.2.} \times \frac{\pi}{2}$$

$$= \frac{21}{32} ka^4 \pi$$

Mass of the lamina $= \frac{21}{32} ka^4 \pi$

c) Solve $(3x+2)^2\frac{d^2y}{dx^2}+5(3x+2)\frac{dy}{dx}-3y = x^2+x+1$

Solution :

Let 3x +2 = v $\quad \frac{dv}{dx}=3$

$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{dx} = 3\frac{dy}{dv}$

$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(3\frac{dy}{dv}\right)=3\frac{d}{dv}\left(\frac{dy}{dv}\right)\frac{dv}{dx}=9\frac{d^2y}{dv^2}$

The given equation changes to,

$$9v^2\frac{d^2y}{dv^2}+15v\frac{dy}{dv}-3y=\left(\frac{v-2}{3}\right)^2+\frac{v-2}{3}+1=\frac{v^2-4v+4}{9}+\frac{v-2}{3}+1$$

Multiplying throughout by 9,

$$81v^2\frac{d^2y}{dv^2}+135v\frac{dy}{dv}-27y=v^2-4v+4+3v-6+9$$

$$81v^2\frac{d^2y}{dv^2}+135v\frac{dy}{dv}-27y=v^2-v+7 \qquad \text{........(1)}$$

Put z = logv $\qquad$ v = e^z

Now, $v\frac{dy}{dv}=Dy,\ v^2\frac{d^2y}{dv^2}=D(D-1)y$

Equation (1) becomes,

[81D(D – 1) + 135D – 27]y = $e^{2z}-e^z+7$

[$81D^2+54D-27$]y = $e^{2z}-e^z+7$

The auxiliary equation is $81D^2+54D-27$= 0.

(D + 1)(D - $\frac{1}{3}$) = 0

D = -1, $\frac{1}{3}$

Complementary Function, C.F. = $c_1e^{-z}+c_2e^{-z/3}$

$= c_1e^{-\log v}+c_2^{-\log v/3}$

$= c_1v^{-1}+c_2v^{-1/3}$

$= c_1(3x+2)^{-1}+c_2(3x+2)^{-1/3}$

Particular Integral, P.I. = $\frac{1}{81\ \ ^2+54D-27}e^{2z}-e^z+7$

$$= \frac{1}{81(2)^2+54(2)-27}e^{2z} - \frac{1}{81(1)^2+54(1)-27}e^{z} + \frac{1}{81(0)^2+54(0)-27}7$$

$$= \frac{e^{2z}}{405} - \frac{e^{z}}{108} + \frac{7}{27}$$

$$= \frac{1}{27}\left(\frac{e^{2z}}{15} - \frac{e^{z}}{4} + 7\right)$$

Resubstituing z = log v

$$= \frac{1}{27}\left(\frac{e^{2logv}}{15} - \frac{e^{logv}}{4} + 7\right)$$

$$= \frac{1}{27}\left(\frac{v^2}{15} - \frac{v}{4} + 7\right)$$

Resubstituing v = 3x + 2

$$\text{P.I.} = \frac{1}{27}\left(\frac{(3x+2)^2}{15} - \frac{(3x+2)}{4} + 7\right)$$

The solution is,

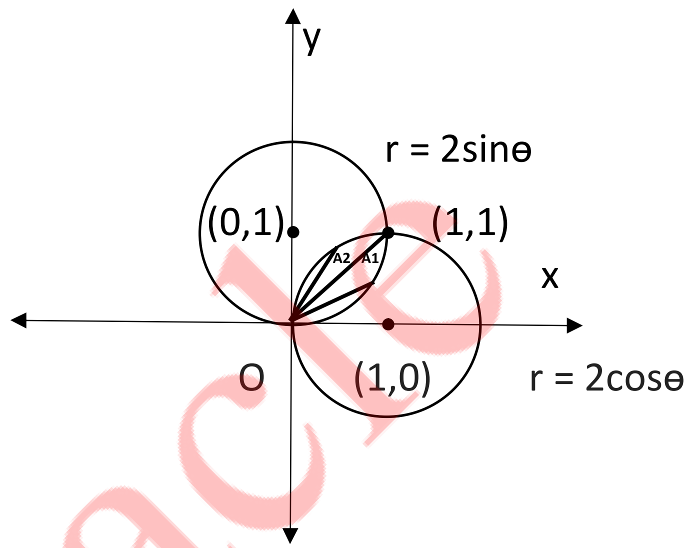
y = C.F. + P.I.

$$\boxed{y = c_1(3x + 2)^{-1} + c_2(3x + 2)^{-1/3} + \frac{1}{27}\left(\frac{(3x+2)^2}{15} - \frac{(3x+2)}{4} + 7\right)}$$

Q.4
a) Find by double integration the area common to the circles r = 2cosθ and r = 2sinθ.

Solution :

We have r = 2cosθ



i.e. $\sqrt{x^2+y^2} = 2\frac{x}{\sqrt{x^2+y^2}}$

$x^2 + y^2 - 2x = 0$

$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$

Centre ≡ (1,0)

Radius = 1

Similarly, r = 2sinθ

i.e. $\sqrt{x^2+y^2} = 2\frac{y}{\sqrt{x^2+y^2}}$

$x^2 + y^2 - 2y = 0$

$x^2 + y^2 - 2y + 1 = 1$

$x^2 + (y-1)^2 = 1$

Centre ≡ (0,1)

Radius = 1

Consider radial strips in both A1 and A2.

In A1, r varies from 0 to 2cosθ and θ varies from 0 to π/4

In A2, r varies from 0 to 2sinθ and θ varies from π/4 to π/2

Area = A1 + A2

$= \int_0^{\pi/4}\int_0^{2cos} r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{2sin\theta} r\,dr\,d\theta$

$= \int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{2cos\theta} d\theta + \int_{\pi/4}^{\pi/2}\left[\frac{r^2}{2}\right]_0^{2sin\theta} d\theta$

$= 2\left[\int_0^{\frac{\pi}{4}}(cos^2\theta)\,d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} sin^2\theta d\theta\right]$

$= 2\int_0^{\frac{\pi}{4}}\frac{cos2\theta}{2}\,d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1-cos2\theta}{2}\,d\theta$

$= \left[\frac{-sin2\theta}{2} + \theta\right]_0^{\frac{\pi}{4}} + \left[\theta + \frac{sin2\theta}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$$= \left(\frac{-\sin\frac{\pi}{2}}{2} + \frac{\pi}{4}\right) + \left(\frac{\pi}{2} + \frac{\sin\pi}{2} - \frac{\pi}{4} - \frac{\sin\frac{\pi}{2}}{2}\right)$$

$$\boxed{\text{Area } = \frac{\pi}{2} - 1}$$

b) Solve $\sin 2x \frac{dy}{dx} = y + \tan x$

<u>Solution</u> :

$$\frac{dy}{dx} - \frac{y}{sin2x} = \frac{tanx}{sin2x}$$

$$\frac{dy}{dx} - \frac{y}{sin2x} = \frac{1}{2cos^2x}$$

Comparing with $\frac{dy}{dx} + P(x)y = f(x)$

P(x) = $-\frac{1}{sin2x}$

f(x) = $\frac{1}{2cos^2x}$

I.F = $e^{\int \frac{-1}{sin} dx}$

= $e^{-\int cosec2xdx}$

= $e^{\frac{-\log(cosec2x - cot\quad)}{2}}$

= $e^{\frac{-\log\left(\frac{1-cos}{sin2}\right)}{2}}$

= $e^{\frac{-\log\left(\frac{2sin^2x}{2sinx.cosx}\right)}{2}}$

= $e^{\frac{-\log(tanx)}{2}}$

I.F. = $\frac{1}{\sqrt{tanx}}$

The solution is,

y x I.F. = $\int P(x). I.F. dx + c$

$\frac{y}{\sqrt{tanx}} = \int \frac{1}{2cos^2x}$ x $\frac{1}{\sqrt{tanx}} dx + c$

$\frac{y}{\sqrt{tanx}} = \int \frac{1}{2co\ ^2x}$ x $\frac{1}{\sqrt{tanx}} dx + c$

$\frac{y}{\sqrt{tanx}} = \frac{1}{2} \int \frac{1}{\sqrt{cos^4x.\frac{sinx}{cosx}}} + c$

$\frac{y}{\sqrt{tanx}} = \frac{1}{2} \int cos^{-3/2}x. sin^{-1/2}xdx + c$

Put $cos^{-\frac{1}{2}}x = t$

$\frac{1}{2} cos^{-3/2}x. sinxdx = dt$

$\frac{1}{2}cos^{-3/2}x.sin^{-1/2}x.sin^{3/2}xdx = dt$

$\frac{1}{2}cos^{-3/2}x.sin^{-1/2}xdx = \frac{dt}{sin^{3/2}x}$ ......(1)

Now,

$cos^{-\frac{1}{2}}x = t$

$t^{-4} = cos^2 x$

$(1 - t^{-4}) = 1 - cos^2 x$

$(1 - t^{-4}) = sin^2 x$

$sin^{3/2}x = (1 - t^{-4})^{3/4}$ .....(2)

Substituting (2) in (1),

$\frac{1}{2}cos^{-3/2}x.sin^{-1/2}xdx = \frac{dt}{(1-t^{-4})^{3/4}}$

$\frac{y}{\sqrt{tanx}} = \frac{1}{2}\int \frac{dt}{(1-t^{-4})^{\frac{3}{4}}} + c$

$\frac{y}{\sqrt{tanx}} = \frac{1}{2}\int \frac{t^3 dt}{(t^4-1)^{3/4}} + c$

Let $t^4 - 1 = g$

$4t^3 dt = dg$

$t^3 dt = \frac{dg}{4}$

$\frac{y}{\sqrt{tanx}} = \frac{1}{2}\int \frac{dg}{4g^{3/4}} + c$

$\frac{y}{\sqrt{tanx}} = \frac{1}{8}\int g^{-3/4}dg + c$

$\frac{y}{\sqrt{tanx}} = \frac{1}{8}\frac{g^{1/3}}{1/3} + c$

$\frac{y}{\sqrt{tanx}} = \frac{3}{8}g^{1/3} + c$

Substituting $g = t^4 - 1$

$\frac{y}{\sqrt{tanx}} = \frac{3}{8}(t^4 - 1)^{1/3} + c$

Substituing $t = cos^{-1/2}x$

$\frac{y}{\sqrt{tanx}} = \frac{3}{8}[(cos^{-\frac{1}{2}}x)^4 - 1]^{1/3} + c$

$$\boxed{\frac{y}{\sqrt{tanx}} = \frac{3}{8}[cos^{-2}x - 1]^{1/3} + c}$$

c) Solve $\frac{dy}{dx} = 3x + y^2$ with initial conditions y_0 = 1, x_0 = 0 at x = 0.2 in steps of h = 0.1 by Runge Kutta method of fourth order.

Solution :

$$\frac{dy}{dx} = 3x + y^2$$

f(x, y) = $3x + y^2, x_0 = 0, y_0 = 1, h = 0.1$

$k_1 = hf(x_0, y_0) = 0.1[3(0) + 1^2] = 0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1\left[3\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1}{2}\right)^2\right] = 0.1252$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1\left[3\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1252}{2}\right)^2\right] = 0.1279$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1[3(0 + 0.1) + (1 + 0.1279)^2] = 0.1572$$

$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = \frac{1}{6}[0.1 + 2(0.1252) + 2(0.1279) + 0.1572]$$

$$k = \frac{1.2634}{6} = 0.2105$$

The approximate value of y at x = 0.2 is = $y_0 + k = 1 + 0.2105 = 1.2105$

Q.5

a) Evaluate $\int_0^1 x^5 \sin^{-1} x\, dx$ and find the value of $\beta\left(\frac{7}{2}, \frac{1}{2}\right)$.

Solution :

Integrating by parts we have,

$$\int_0^1 x^5 \sin^{-1} x\, dx = \left[\sin^{-1} x . \frac{x^6}{6}\right]_0^1 - \int_0^1 \frac{x^6}{6} . \frac{1}{\sqrt{1-x^2}} dx$$

$$\int_0^1 x^5 \sin^{-1} x\, dx = \frac{\pi}{2} . \frac{1}{6} - \frac{1}{6}\int_0^1 \frac{x^6}{\sqrt{1-x^2}} dx$$

Put x = sinɵ   dx = cosɵdɵ

$$I = \frac{\pi}{12} - \frac{1}{6}\int_0^{\pi/2} \frac{\sin^6\theta}{\cos\theta} \cos\theta d\theta$$

$$= \frac{\pi}{12} - \frac{1}{6}\int_0^{\pi/2} \sin^6\theta d\theta$$

$$= \frac{\pi}{12} - \frac{1}{6} . \frac{5}{6} . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}$$

$$= \frac{\pi}{12} - \frac{5\pi}{192}$$

$$I = \frac{11\pi}{192}$$

$$\boxed{\int_0^1 x^5 \sin^{-1} x\, dx = \frac{11\pi}{192}}$$

$$\boxed{\beta\left(\frac{7}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{7}{2}\right).\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}+\frac{1}{2}\right)} = \frac{\Gamma\left(\frac{7}{2}\right).\Gamma\left(\frac{1}{2}\right)}{\Gamma(4)} = \frac{\frac{7 \times 5 \times 3}{2 \times 2 \times 2}\Gamma\left(\frac{1}{2}\right).\Gamma\left(\frac{1}{2}\right)}{3!} = \frac{35}{16}\Gamma^2\left(\frac{1}{2}\right)}$$

b) The differential equation of a moving body opposed by a force per unit mass of value cx and resistance per unit mass of value bv^2 where x and v are the displacement and velocity of the particle at that time is given by $v\frac{dv}{dx} = -cx - bv^2$. Find the velocity of the particle in terms of x, if it starts from rest.

Solution :

We have $v\frac{dv}{dx} = -cx - bv^2$

Putting $v^2 = y, v\frac{dv}{dx} = \frac{1}{2}\frac{dy}{dx}$

$\frac{1}{2}\frac{dy}{dx} + by = -cx$

$\frac{dy}{dx} + 2by = -2cx$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

I.F. = $e^{\int Pdx} = e^{\int 2bdx} = e^{2b}$

The solution is $ye^{2bx} = \int e^{2bx}(-2cx)dx + c'$

$ye^{2bx} = -2c\int xe^{2bx}dx + c'$

$ye^{2bx} = -2c\left(x\frac{e^{2bx}}{2b} - \int 1.\frac{e^{2bx}}{2b}dx\right) + c'$

$ye^{2bx} = -2c\left(x\frac{e^{2bx}}{2b} - \frac{e^{2bx}}{4b^2}\right) + c'$

Resubstituting $y = v^2$

$v^2e^{2bx} = -\frac{cx}{b}e^{2bx} + \frac{c}{2b^2}e^{2b} + c'$

By data, when x = 0, v = 0 So, $c' = -\frac{c}{2b^2}$

$v^2e^{2bx} = -\frac{cx}{b}e^{2bx} + \frac{c}{2b^2}e^{2bx} - \frac{c}{2b^2}$

$\boxed{v^2 = \frac{c}{2b^2}(e^{2bx} - 1) - \frac{cx}{b}}$

c) Evaluate $\int_0^6 \frac{dx}{1+4x}$ by using i) Trapezoidal ii) Simpsons (1/3)rd and iii) Simpsons (3/8)th rule.

Solution :

Dividing the interval to 6 parts by taking each subinterval equal to $h = \frac{6-0}{6} = 1$

x	0	1	2	3	4	5	6
$y = \frac{1}{1+4x}$	1	$\frac{1}{5}$	$\frac{1}{9}$	$\frac{1}{13}$	$\frac{1}{17}$	$\frac{1}{21}$	$\frac{1}{25}$
Ordinate	y_0	y_1	y_2	y_3	y_4	y_5	y_6

i) By Trapezoildal Rule,

$I = \frac{h}{2}[X + 2R]$

Now, X = sum of the extremes = $1 + \frac{1}{25} = 1.04$

And, R = sum of the remaining = $\frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \frac{1}{17} + \frac{1}{21} = 0.4944$

$$\boxed{I = \frac{h}{2}[X + 2R] = \frac{1}{2}[1.04 + 0.4944] = 0.7672}$$

ii) By Simpsons (1/3)rd rule,

$I = \frac{h}{3}[X + 2E + 4O]$

Now, X = sum of the extremes = $1 + \frac{1}{25} = 1.04$

2E = 2 x sum of the even ordinates = $2\left(\frac{1}{9} + \frac{1}{17}\right) = 0.3398$

4O = 4 x sum of the odd ordinates = $4\left(\frac{1}{5} + \frac{1}{13} + \frac{1}{21}\right) = 1.2981$

$$\boxed{I = \frac{h}{3}[X + 2E + 4O] = \frac{1}{3}[1.04 + 0.3398 + 1.2981] = 0.8926}$$

iii) By Simpsons (3/8)th rule,

$I = \frac{3h}{8}[X + 2T + 3R]$

Now, X = sum of the extremes = $1 + \frac{1}{25} = 1.04$

2T = 2 x sum of the multiples of 3 = $2 \times \frac{1}{13} = 0.1538$

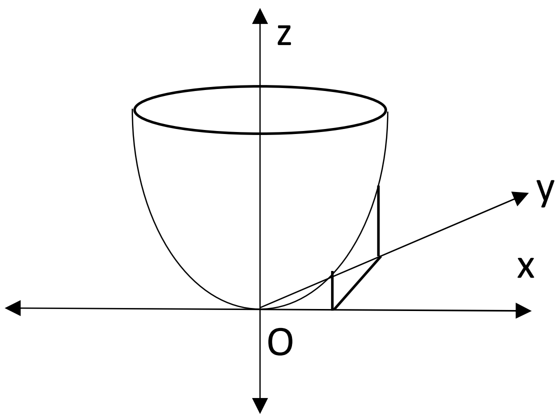
3R = 3 x sum of the remaining = $3\left(\frac{1}{5} + \frac{1}{9} + \frac{1}{17} + \frac{1}{21}\right) = 1.2526$

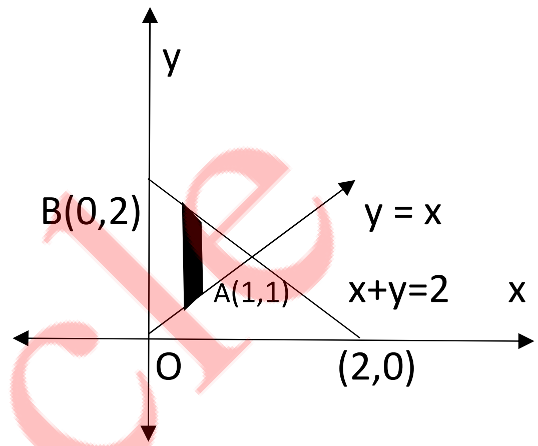
$$\boxed{I = \frac{3h}{8}[X + 2T + 3R] = \frac{3}{8}[1.04 + 0.1538 + 1.2526] = 0.9174}$$

Q.6

a) Find the volume of the region that lies under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines y = x, x = 0 and x + y = 2 in the xy plane.

Solution :





The base of the required solid is a triangle OAB.
Take a strip parallel to the y-axis from y = x to y = 2-x. The strip moves parallel to itself from x = 0 to x = 1. Z varies from 0 to $x^2 + y^2$.

$$V = \int_0^1 \int_x^{2-x} \int_0^{x^2+y^2} dz dy dx = \int_0^1 \int_x^{2-x} (x^2 + y^2) dy dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{2-x} dx$$

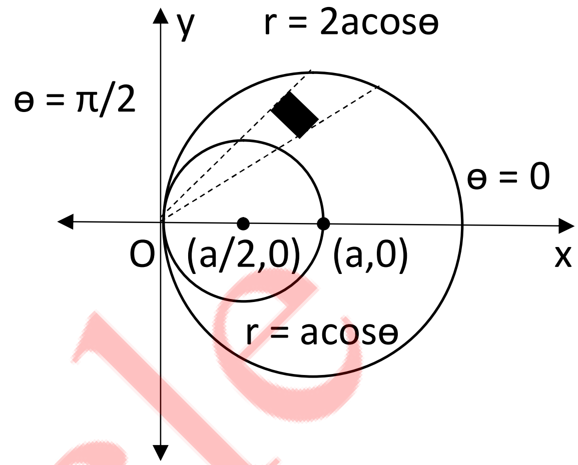
$$= \int_0^1 \left[x^2(2-x) + \frac{(2-x)^3}{3} - x^3 - \frac{x^3}{3}\right] dx = \int_0^1 2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3} dx$$

$$= \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12}\right]_0^1 = \frac{2}{3} - \frac{7}{12} - \frac{1}{12} + \frac{16}{12} = \frac{4}{3}$$

$$\boxed{V = \frac{4}{3}}$$

b) Change to polar coordinates and evaluate $\iint y^2 dxdy$ over the area outside $x^2 + y^2 - ax = 0$ and inside $x^2 + y^2 - 2ax = 0$

Solution :



$$x^2 + y^2 - ax = 0$$

$$x^2 - ax + \left(\frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$$

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$$

Centre ≡ (a/2,0)

Radius = a/2

And,

$$x^2 + y^2 - 2ax = 0$$

$$x^2 - 2ax + a^2 + y^2 = a^2$$

$$(x - a)^2 + y^2 = a^2$$

Centre ≡ (a,0)

Radius = a

Putting x=rcosɵ and y=rsinɵ in $x^2 + y^2 - ax = 0$ we get $r^2 = arcos\theta$ i.e. $r = acos\theta$ and in $x^2 + y^2 - 2ax = 0$ we get $r^2 = 2arcos\theta$ i.e. $r = 2acos\theta$

Considering a radial strip, r varies from acosɵ to 2acosɵ and ɵ varies from 0 to $\frac{\pi}{2}$.

I = $\iint y^2 dxdy$

I = $2\int_0^{\frac{\pi}{2}} \int_{acos\theta}^{2acos\theta} (rsin\theta)^2 rdrd\theta$

I = $2\int_0^{\frac{\pi}{2}} \int_{acos\theta}^{2acos\theta} r^3 sin^2\theta drd\theta$

I = $2\int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_{acos\theta}^{2acos\theta} sin^2\theta d\theta$

I = $\frac{1}{2}\int_0^{\frac{\pi}{2}} (16a^4cos^4\theta - a^4cos^4\theta)\, sin^2\theta d\theta$

I = $\frac{15a^4}{2}\int_0^{\frac{\pi}{2}} cos^4\theta sin^2\theta d\theta$

I = $\frac{15a^4}{2}$ x $\frac{3.1.1}{6.4.2}$ x $\frac{\pi}{2}$

$$\boxed{I = \frac{15\pi a^4}{64}}$$

c) Solve by method of variation of parameters

$$\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$$

Solution :

The auxiliary equation is $D^2 + 1 = 0$

$D = i, -i$

Complementary Function, C.F. = $c_1\cos x + c_2\sin x$

Here $y_1 = \cos x$, $y_2 = \sin x$ and $X = \frac{1}{1+\sin x}$

Let Particular Integral, P.I = $uy_1 + vy_2$

Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} cosx & sinx \\ -sinx & cosx \end{vmatrix} = cos^2x + sin^2x = 1$$

$$u = -\int \frac{y_2 X}{W} dx = -\int \frac{sinx}{1} \text{ x } \frac{1}{1+sinx} dx = -\int \frac{sinx}{1-sinx} \text{ x } \frac{1-sin}{1+sinx} dx = -\int \frac{sinx - sin^2x}{cos^2x} dx$$

$$= -\int (secx.tanx - tan^2x)\, dx = -\int (secx.tanx - sec^2x + 1)dx$$

$$u = -secx + tanx - x$$

$$v = \int \frac{y_1 X}{W} dx = \int \frac{cosx}{1} \text{ x } \frac{1}{1+s} dx = \log(1 + \sin x)$$

The complete solution is,

y = C.F. + P.I.

$$\boxed{y = c_1\cos x + c_2\sin x + \cos x(-secx + tanx - x) + \sin x.\log(1 + \sin x)}$$